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The Role of Quantification Operators in the Development of Conservation of Quantity[1]

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An analysis of the quantitative processes underlying conservation of quantity is presented. Models of quantitative operators (subitizing, counting, estimation) are derived from adult performance in quantification tasks, and some features of the operators are described. The emergence of conservation is described in terms of the development of the operators and a set of rules which evoke them and coordinate their results. Empirical data related to the developmental argument is discussed.

The classic version of the Piagetian test for conservation of quantity starts with the presentation of two distinct collections of equal amounts of material (e.g., two rows of beads, two vessels of liquid, two lumps of clay, etc.). First the child is encouraged to establish their quantitative equality (e.g., "Is there as much to drink in this one as in that one?"; "Is it fair to give this bunch to you and that bunch to me?"; etc.). Then he observes one of the collections undergo a transformation that changes some of its perceptual features while maintaining its quantity (e.g., stretching, compressing, pouring into a vessel of different dimensions, etc.). Finally, the child is asked to judge the relative quantity of the two collections after the transformation. To be classified as "having conservation" the child must be able to assert the continuing quantitative equality of the two collections without resorting to a requantification and comparison after the transformation. That is, his response must be based not upon another direct observation, but rather upon recognition of the

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logical necessity for initially equal amounts to remain equal under "mere" perceptual transformations.

This paper presents a developmental model of the processes that underlie successful performance on conservation tasks. The model is motivated by the observation that conservation requires the child to make paired comparisons of quantity, using processes that encode quantitative features and produce quantitative representations. These processes are called *quantification operators.* Our central thesis is that the development of conservation is dependent upon the emergence of different quantification operators.

The developmental theory is predicated upon two general systemic principles.

(a) *The developing system constantly searches for consistent sequences which enable it to eliminate redundant processing. A consistent sequence is an internal representation of environmental inputs and system processes that always yield the same result. It is not simply an environmental regularity, but rather regularity arising from the interaction between the environment and the system.*

A typical sequence might consist of an initial knowledge state (Newell and Simon, 1972), an operation upon that state, and a final test of the result of the operation. If the environment were such that a specific state–operator–test sequence repeatedly produced the same result, then the final test would become redundant. The redundancy elimination principle ultimately leads the system to dispense with the final test, and instead to retrieve from long-term memory (LTM) the result of previous test applications for that specific state–operator sequence. This is tantamount to prediction of what the outcome would have been if the test had actually been made. At some intermediate stage between the initial appearance of the LTM representation of the sequence and total reliance upon it, the system uses both modes of operation. The LTM representation is used as a predictor and the actual test is carried out to verify that prediction.

(b) *If, in a particular context, the system is unable to detect consistent sequences, it widens the basis of its search. In the case of the development of quantity conservation this is accomplished by widening the range of dimensions under consideration in the search for consistent indicators of quantity.*

The paper is divided into three sections. In the first section we define three types of quantification operators, drawing upon the evidence from adult performance on quantification tasks. An account of the development of these operators is also presented. In the second section we describe the properties of a system that "has" conservation of discontinuous quantity (e.g., beads). The description takes the form of a set of rules that evoke quantification operators and coordinate their outputs. The third section describes the development of the system from a stage in which it has neither quantification operators nor conservation rules, through the discontinuous quantity stage described in the second section, and on to a further stage in which it has conservation of continuous quantity (e.g., area, volume). The emerging system has a bootstrapping nature: the development of one quantification operator facilitates the establishment of rules that in turn facilitate the development of another quantification operator.

QUANTIFICATION OPERATORS

A quantification operator is an organized collection of elementary processes that takes as input the stimulus to be quantified (e.g., a collection of blocks) as well as specified constraints (e.g., red only) and produces as output a quantitative symbol. Quantitative symbols are labeled internal representations (e.g., "two," "long," "tiny") that can be used in quantitative comparisons. Given two such symbols, the organism can determine their relative magnitudes, whereas given two nonquantitative symbols, it can determine only whether or not they are identical.

We postulate the existence of three quantification operators: subitizing, counting, and estimation. Evidence for these operators comes from analyses of reaction times (RT) and errors in tasks requiring adult subjects to report the number of items in a display (Jensen, Reese, & Reese, 1950; Kaufman, Lord, Reese, & Volkman, 1949; Saltzman & Garner, 1948; Taves, 1941; Woodworth & Schlosberg, 1954, pp. 90–105). A reanalysis of these studies and some new evidence is provided in Klahr (1973). The earlier investigations were addressed to the question of whether the time required to quantify a collection is independent of the number of items viewed (n). The answer appears to be negative. A plot of RT versus n yields a monotone increasing curve in the range from 1 to 30 items. However, around $n = 5$ the slope abruptly changes from approximately 40 to approximately 300 milliseconds (ms) per item (Klahr, 1973). There are corresponding discontinuities in error rates and reported self-confidence. Finally, there is a subjectively different experience for n above and below 5. (The immediacy of judgments for small n may account for the erroneous impressions of early investigators, e.g., Jevons, 1871, that their RTs were independent of n.)

Subitizing

We have retained the term "subitizing" (Kaufman *et al.*, 1949) for the operator used to quantify small collections. The parameters that define subitizing (Q_s) are a slope of 40 ms and a maximum range of five or six items. These parameters are similar to short-term memory (STM) scanning rates (Sternberg, 1966, 1967) and capacity limits (Miller, 1956). Thus, they suggest that subitizing involves a serial self-terminating scan of STM for a match between the encoded stimulus and a short ordered list of quantitative symbols. We assume that adults have, stored in long-term memory (LTM), a *subitizing list* consisting of an ordered set of quantitative symbols representing the first five or six cardinal values. The quantitative symbols on the subitizing list are distinct representations of pure cardinality (e.g., "twoness").[2]

When Q_s is invoked, the subitizing list is transferred from LTM to STM and scanned at approximately 40 ms per symbol.[3] If a match is found between the encoded stimulus and one of the symbols on the subitizing list, the search is terminated and the label associated with the matching

[2] A more formal statement of the "number essence" of these symbols utilizes the notion of a *tolerance space* (Zeeman & Buneman, 1968). If, in the internal representation of the quantitative aspects of the stimulus, two objects a and b are not distinguished, then they are *within tolerance,* $(a \sim b)$. Otherwise, they are *outside tolerance*. The tolerance space (TS) is defined as the set of pairs $[a, b]$ such that $a \sim b$. The tolerance is a function of both the stimulus and the goal of the quantification effort. For example, if the goal is to determine the number of pages in a book, then the letters, words, and paragraphs on a given page are not distinguished, and all the pairs of such elements on a given page are within tolerance. Thus each page is a single TS. Similarly, if the goal were to determine the number of words in the book, each word would constitute a TS. A TS is represented by the primitive quantitative element in the system: a tolerance space atom (TSA). We assume that the quantitative symbols on the subitizing list are sublists of one, two, three, etc., TSAs. The encoded representation of the stimulus is also a list of TSAs produced by an interaction between the stimulus and the system. Early in the processing sequence, perhaps in iconic memory (Neisser, 1967; Sperling, 1960), the tolerance of the stimulus to be encoded is determined according to the goal of the quantification attempt. We assume, as do most of the STM scanning models that the time for comparison of any specific item in STM with the test item is constant. Thus, although the quantitative symbols on the subitizing list are sublists of differing length, their match with the test stimulus can be viewed as essentially parallel, at least with respect to the scanning rate.

[3] The most often quoted results from Sternberg's studies are slopes of RT as a function of the length of the list being scanned, i.e., the "length functions" (Sternberg, 1967). In our model the length of the postulated subitizing list is fixed, and the comparable Sternberg slopes are the "position functions": RT vs serial position. For context recognition Sternberg reports a mean over subjects of 92 ms per item with individual subject means ranging from 22 to 240. However, Ellis and Chase (1971, p. 384) report a 40 ms slope for the position function in item recognition tasks.

symbol is produced. If no match is found, then another quantification operator is applied (either counting or estimation).

Counting

The counting operator (Q_c) produces the 300 ms slope described above. Q_c requires the coordination of processes that notice each object while generating the sequence of number names. When there are no more objects to be noticed, the current name is assigned to the collection of objects.

Under some conditions (e.g., instructions for speed) subjects appear to count by subitizing two or three items and adding the result (Klahr, 1973). However, we define counting here as the one-at-a-time process that subjects utilize when so instructed. Q_c thus requires two auxiliary structures in LTM: a finite, ordered list of number names together with some rules for generating number names indefinitely, and rules to ensure that each object is noticed only once. There are several forms of such attention directing processes, ranging from motor systems that move objects as they are noticed, to well-defined eye movement patterns.

More than half of the additional processing time per item appears to come from moving through the number name list. Beckwith and Restle (1966) gave explicit instructions to "enumerate as quickly as possible" (p. 439) and found average slopes of 350 ms. In a task requiring subjects to implicitly recite the alphabet from an initial letter to a final letter, Olshavsky and Gregg (1970) found a processing rate of 150 ms per item, similar to the rate of implicit recitation found by Landauer (1962). When subjects were required to scan a specified number of letters the rate decreased to 260 ms. Direct evidence for the rate of "spatial enumeration without counting" (Potter & Levy, 1968) is not available.

Estimation

The logical necessity for a quantifier other than subitizing and counting is clear. Quantitative symbols can be produced in situations involving great numbers, or limited exposure duration, or continuous quantity, where neither Q_s or Q_c could function. For example, Kaufman *et al.* (1949) found that with a 200 ms exposure of from 1 to 200 dots, RT was constant above $n = 6$.

The determination of the nature of the estimation quantifier (Q_e) is puzzling, because we are attempting to characterize a system that can encode continuous quantity and yet (if it is to be consistent with the body of literature on information processing models of cognition) must be composed of discontinuous data structures and processes. Q_e is essentially measurement: repeated application of a "standard" unit. These standards are based upon the idiosyncratic experience of the particular system. Thus, some people estimate length in terms of football fields, others in terms of automobiles. In any given system many such standards exist for each class of quantity (length, number, volume, etc.) and they are utilized according to the particular task requirements.[4]

Development of Quantification Operators

In adults, Q_s, Q_c, and Q_e are fully developed. For very small n, Q_s is used, for very large n, or for continuous quantity, Q_e is used; for intermediate situations Q_c is used.

In this section we shall outline some features of the relative developmental rates for these operators. Two kinds of things may change as an information processing system develops: its processes and its representations. Here we focus upon representational changes, deferring an account of process changes until the section on development of conservation rules.

Representations for quantity. What might be the steps leading to the acquisition of the postulated subitizing list? We postulate the emergence of an increasingly general and efficient representation. The representations move from the concrete, heterogenous to the abstract, homogenous. At the first stage we will consider, the organism represents a collection of two identical objects by two identical symbol structures. Thus, a pair of identical items, A and B, are represented as shown in Fig. 1a. The total collection X consists of symbol structures for both A and B. The symbol structures themselves consist of some elementary symbols (α, β, γ) that completely characterize the objects. Such a representation is highly redundant, and the system, through its efforts at redundancy elimination, eventually adopts an alternative representation. In dealing with collections of identical objects, only a single fully described symbol structure is created, and the other objects are represented by special symbols (ϕ) indicating replications of the symbol structure to which they are attached.[5] Figure 1b shows this representation for a collection, X, of three identical objects.

[4] The zero slope for estimation would result from a Q_e in which the size of the standard is itself a function of n. For example, for $n \leq 50$ every 10 dots might be considered to be a TS. For $50 < n \leq 200$ 25 dots might be a TS, etc. If the rate of Q_e was about the same as Q_s, once the tolerance had been determined, then the plot of RT versus n would be a shallow sawtooth. If we allow some overlap on the ranges that determine the TS size then we would find the zero slope with increasing dispersion reported by Kaufman *et al.* (1949).

[5] Newell and Simon (1972) use the terms "symbol type" for fully described object and "symbol token" for the representation of distinct instances of the symbol type. Our model starts by representing a collection of items as several identical symbol types (Fig. 1a). The second stage (Fig. 1b) utilizes a special kind of token, one that denotes replication of whatever it is attached to (as in the use of the ditto symbol).

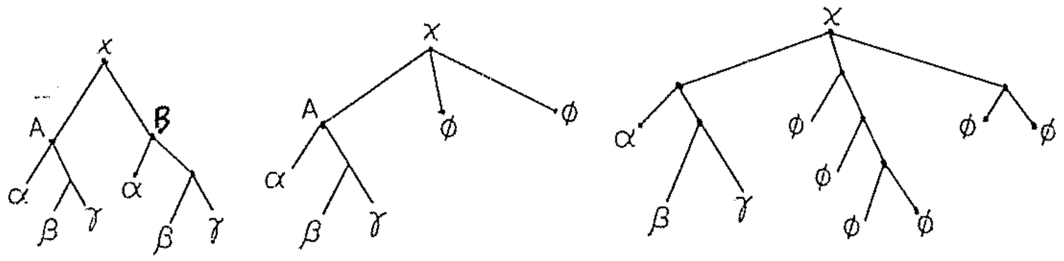

FIG. 1. Development of increasingly efficient quantitative representations. Left, a; center, b; right, c.

With increasing experience in dealing with a particular type of object, several such "model collections," (Dantzig, 1954) each representing collections of different numerosity, are constructed and stored in LTM. Figure 1c shows such a structure for 1, 4, and 2 A's. A similar process occurs for other types of objects with which the system has to deal regularly. Ultimately the system has a number of lists (as yet not numerically ordered) representing the same range of cardinal numbers but tied by labeling to specific classes of objects.

The emergence of the next phase in the refinement of the subitizing list is again attributed to the removal of redundancy. The number of redundant lists is steadily reduced by "broadening" the labels (i.e., reducing the specificity of the fully described symbol structure) to permit the use of single collections of lists in coping with several classes or categories.[6] This movement away from several lists linked to specific, concrete classes is eventually complete and gives rise to a single, abstract system of sublists of tolerance space symbols.

Some developmental evidence relevant to this process has been provided by Gast (1957) in an investigation of the degree of perceptual and functional heterogeneity of stimulus materials which a child can tolerate in admitting a collection of objects as representing a single cardinal set for purposes of enumeration or number matching. His findings indicate an initial stage in which virtually complete homogeneity of the elements is required, a second in which perceptual diversity is possible, within certain limits of qualitative resemblance among the elements, and a final one in which the objects may belong to several disjunctive and altogether disparate classes.

How do the symbols representing different cardinal values become labeled and ordered? There seem to be two likely processes. In one, the familiar number names become attached to the appropriate subitizing

[6] A fairly advanced point in this process is illustrated in the Tsimshian language of a British Columbian tribe (Dantzig, 1954). It comprises seven distinct sets of number words one of which is used in dealing with flat objects, one with round objects, another for long objects and so on.

symbols by direct association. A child matches a collection with the appropriate symbol and an adult verbalizes the appropriate label. Alternatively, the labels may be derived from an interaction between subitizing and counting. Once a child has learned the sequence of number names and combined them with the attention direction techniques required for the process of counting, he has two systems available for quantifying collections in the subitizing range. When counting techniques are being learned, small collections are both subitized and then counted. This facilitates the association of number labels with the ordinal symbols on the subitizing list. Once the subitizing symbols have been so labeled, they are rearranged to bring them into correspondence with the order of the list of familiar symbols used by counting.

The components of counting are learned at different rates. Descoeudres (1921) found that it was not until the age of 4:6 yr, on average, that children can accurately count collections of up to six objects even when placing their fingers on each object as it is counted. Learning of the sequence of number names is considerably quicker. For example, children are able to correctly generate the name sequence from 1 to 4 at the age of 2:6 but it is considerably longer before the coordination with the noticing techniques is achieved and the semantic basis for counting established.

Developmental sequence for subitizing and counting. We have alluded several times to the relation between the development of Q_s and Q_c. In this section we shall address the issue directly. Flavell (1971) has convincingly argued that the concept of developmental primacy is ill-defined without simultaneous consideration of relative starting points, growth rates, and points of "functional maturity." In adults both Q_s and Q_c are fully developed and utilized under appropriate conditions. Thus, they are related, in Flavell's (1972) terms by "addition": once available "both continue to be used for the remainder of one's cognitive career" (p. 287). However, the development of Q_c rests upon the emergence of the initial quantitative representations utilized by Q_s, thus the operators are also related by "mediation." Flavell describes the mediation of item X_2 by item X_1 as follows:

> *The acquisition of X_1 could be described as constituting some sort of developmental route or path to X_2, as providing an occasion or opportunity for the emergence of X_2, as facilitating the genesis of X_2 . . .* (p. 312).

The interdependence between Q_s and Q_c is more complex than any of Flavell's ideal sequences, since Q_c, once it has started to develop, influences the representation used by Q_s. As described earlier, the ordering of the subitizing list is partly accomplished through the simultaneous application of Q_s and Q_c to small collections.

On logical grounds a Q_s of limited capacity (say $n = 2$) must develop before Q_c. If Q_s is applied to a static collection, the quantitative symbol produced at one moment is the same as that produced at the next. For the emerging system this stability does not yet exist for Q_c. The unique topological properties of collections of small n, and the invariance of these properties over a wide range of perceptual differences contribute to the early emergence of Q_s as a reliable quantifier. This is not to say that the other operators are not employed, but just that they do not yet yield any consistent results, even in a static environment. Q_c requires a socially transmitted technology including verbal labels, noticing orders, and place keeping—processes that develop more slowly than subitizing. (Q_e at this early stage of development is the source of much inconsistency, because, as we shall see below, there is not yet a system of rules that considers the relationships between the various quantities that can be estimated—density, length, width, etc.—and the total amount being quantified.) Furthermore, it is impossible for a system that has not yet developed representations for numerosity to attach the number labels of counting to anything meaningful. Number discrimination must precede counting, and the first few such symbols constitute the early emergence of Q_s.

Thus, we postulate the onset of Q_s prior to the onset of Q_c, and a growth period during which the upper range of Q_s increases from $n = 1$ or 2 to $n = 5$ or 6, while the range of Q_c is extended indefinitely. A plot of maximum range (n) versus age would show a curve for Q_s that started at the origin and asymptotically approached the upper limit, and a curve for Q_c that started during the second year and increased, perhaps positively accelerated, indefinitely. These two curves would cross in the region of $n = 2$–3, and age = 2–3 yr.

Our operational definition of Q_s is based upon the reaction times for verbal responses. There is little direct empirical data from very young children bearing upon the postulated developmental interactions. Gelman (1972b), in a comprehensive review of empirical investigations of early number concepts, compares results from Beckmann (1924) and Descoeudres (1921) and her own investigations (Gelman, 1972a) in which subitizing is defined by the *absence* of overt counting. By this criterion, for n between 3 and 6, children are more likely to count than to subitize the younger they are (age range 4–6 yr). For any given age, the frequency of counting increases with n. From this and other similar studies Gelman concludes that counting precedes subitizing.

There seem to be three problems with this position.[7] First, the definition and scoring of counting versus subitizing appears to both under- and overestimate the amount of subitizing. It underestimates subitizing when *both* Q_s and Q_c are used. As indicated earlier, during the development of Q_c, both Q_s and Q_c are often used on the same task. It overestimates subitizing (as defined by us) when more than 60% of the 6-yr olds are reported as subitizers for $n = 6$. Adult studies tend to show an upper range for Q_s of 4 or 5. Second, it does not account for the fact that when $n = 2$, subitizing is almost always used (75% of the time for 4-yr olds, 98% for 6-yr olds). Finally, it does not offer any explanation for the process whereby quantification, at least to the extent of number discrimination, could take place in a nonverbal system, e.g., birds (Koehler, 1949).

CONSERVATION OF QUANTITY

In this section we formally describe the principal processes in a system that has conservation of quantity.

Let $x, y, z \equiv$ Internal symbols representing collections of material;
$x', y', z' \equiv$ representations for collections of material after any transformation;
$Q_i \equiv$ any quantification operator i, $\epsilon\{s,c,e\}$, where s-subitizing, c-counting, e-estimation;
$x_i \equiv$ quantitative symbol for collection x produced by operator Q_i;
$T_\pm \equiv$ addition/subtraction transformation;
$T_p \equiv$ perceptual transformation (the class of all $T_\pm$ and the class of all T_p are mutually exclusive);
$= \ \equiv$ same quantitative symbol;
$\underset{=}{Q} \equiv$ equal quantity.

There are three kinds of dynamics for which we need additional notation:

i. $T(x) \rightarrow x'$ internal representation for the application of a transformation to an external collection.
ii. $Q_i(x) \rightarrow x_i$ denotes the application of an operator to a collection; it produces a quantitative symbol.
iii. $C \rightarrow\rightarrow A$ is a *production rule* consisting of a *condition* (C) and an *action* (A). The condition is a set of tests for knowledge

[7] Although we disagree with Gelman on this point, there are many similarities between Gelman's (1972b) theoretical position and ours. There are a few terminological differences. She uses "estimator" for what we call quantification operator, and "operator" for what we call the conservation rules. Thus, Gelman describes subitizing and counting as two kinds of estimators.

elements, i.e., symbols representing the current state of knowledge of the system; the action indicates what new elements are to be added to the knowledge state if all the elements in C are satisfied. (For a comprehensive and definitive statement on the use of production rules as representations for problem solving systems see Newell & Simon, 1972; applications to cognitive development are presented in Klahr & Wallace, 1972.)

There is a major variant of the conservation task in which only a single collection is used. There is no initial comparison, and the posttransformation judgment is based upon the relative quantity in the initial and final forms of the collection. We use Elkind's (1967) terminology in which the two-collection task is called *equivalence conservation* (EC) and the one-collection task is *identity conservation* (IC).

As perceived by a system that has conservation, EC can be represented as follows:

i. Two collections are quantified:

$$Q_i(x) \rightarrow x_i;\ Q_i(y) \rightarrow y_i.$$

ii. Their quantitative equality is established. This requires the application of a rule that says, in effect "if the quantitative symbols for two collections are the same, then those collections are of the same quantity." This can be expressed as a production rule:

$$(x_i = y_i) \longrightarrow (x \overset{Q}{=} y).$$

iii. Collection y undergoes a perceptual transformation:

$$T_p(y) \rightarrow (y').$$

iv. The relative amount of the two collections is determined by applying the conservation rule:

$$(x \overset{Q}{=} y)(T_p(y) \rightarrow y') \longrightarrow (x \overset{Q}{=} y')$$

This rule says: if you know that collections x and y were of equal quantity, and that y underwent a perceptual transformation which changed it to y', then you also know that collections x and y' are of equal quantity. Since the two elements that appear as conditions in this rule have been previously entered into the knowledge state, the condition is satisfied. Thus, the action is taken: the fact

$(x \overset{Q}{=} y')$ is added to the knowledge state.

The representation for IC is much simpler, since it involves only the initial and final forms of a single collection. The sequence is:

i. Transform collection: $T_p(x) \rightarrow x'$,

ii. Apply IC rule: $(T_p(x) \rightarrow x') \longrightarrow (x' \overset{Q}{=} x)$

THE DEVELOPMENT OF CONSERVATION

A developmental theory of conservation must account for the emergence and coordination of the functions utilized above. It must describe the process whereby addition/subtraction transformations acquire a special status that is related to quantity. The development of discriminations between T_p and $T_{\pm}$ is central to the concept of quantity conservation. Finally, it must describe the process whereby a system that has reliable means both of determining quantity and of discriminating between transformations eventually develops the conservation inference: "nothing added, nothing subtracted means equal quantities remain equal."

In this section we will offer an account of conservation based upon the emergence of Q_s, Q_c, and Q_e as reliable quantifiers. Our argument is that Q_s enables the system to first develop rules for IC with discontinuous quantity. From this development follow extensions to EC of number and continuous quantity and to conservation of inequality as well as equality.

Identity Conservation Based Upon Subitizing

The first stability developed is based upon the consistent results yielded by Q_s. The end result of the development of Q_s (described earlier) is its status as a reliable indicator of quantity. This is represented by three production rules:

$$(x_s = y_s) \longrightarrow (x \overset{Q}{=} y) \quad (1)$$

$$(x_s < y_s) \longrightarrow (x \overset{Q}{<} y) \quad (2)$$

$$(x_s > y_s) \longrightarrow (x \overset{Q}{>} y). \quad (3)$$

[Rules 2 and 3 introduce notation for inequality of symbols ($<$, $>$) and for corresponding quantitative inequality of collections ($\overset{Q}{<}$, $\overset{Q}{>}$).] The function of Rule 1, for example, is to scan the symbols representing the outcome of a comparison of two quantitative symbols ($=$) and to produce the appropriate quantitative relational attribute ($\overset{Q}{=}$). This can then be added to the representations for the extensive properties of the collections being compared. Thus, the system can represent the fact that collection x, in addition to being red, etc., is equal in quantity to collection y. We can view these rules as explicit statements that symbols produced by Q_s are what Wallach (1969) calls "indicator properties": "perceptible properties, sameness of which indicates equality and difference inequality (p. 207)." Q_s provides the basis for the emergence of the first such indicator properties.

Over time, the system observes many transformations applied to small collections, as the result of its own actions or those of some external agent. In encoding these situations it builds lists of initial state–transformation–final relation sequences and stores them in LTM. Sequences produced by the application of Q_s, Q_c, or Q_e to specific situations are stored. At first no operator provides a basis for the discrimination of quantity preserving transformations from quantity modifying transformations. The emergence of Q_s as the first reliable indicator of quantity facilitates this initial differentiation. The many naturally occurring forms of the identity conservation paradigm provide the necessary experience for this learning. On innumerable occasions (e.g., handling blocks, dolls, cookies) the child quantifies a small collection with Q_s, observes a transformation, quantifies the resultant collection and compares the two quantitative symbols. Some transformations consistently yield the result that $x_s = x'_s$. Others yield $x_s > x'_s$, and still others $x_s < x'_s$. The ability to cope with all three relations is not attained simultaneously. Equality and inequality seem to develop sequentially, although the experimental evidence on their order of appearance is inconclusive (Beilin, 1968). Within the inequality relation there is some indication that "more" is coped with successfully before "less" (Donaldson & Wales, 1970). These consistent sequences become classified, in LTM, into three distinct classes based upon this relational test because it is the only quantitative regularity that exists at this point. What were initially discriminable but arbitrarily labeled transformations become classified according to the relation they produce between x_s and x'_s. Thus certain transformations (e.g., rotations, compressions) are classified as those for which Q_s consistently yields the same result after the transformation as before. Other transformations are associated with the consistent production of either greater (e.g., transfer from parent's hand to collection) or lesser quantities (e.g., placing in mouth and swallowing).

This three-way classification defines the transformations. When a transformation is observed, its class, denoted by the relation it produces between x_s and x'_s is determined. Concurrently with this development, the rules that infer quantitative relations from relations between symbols produced by Q_s (Rules 1–3) are also developing. Thus the transformations themselves become known to the system as either quantity preserving (T_p) or as quantity changing ($T_{\pm}$).

According to the general system principle these regularities are initially used to predict the outcome which is then verified by executing the final test. In their final form, the test is omitted and the system relies entirely upon three rules for identity conservation:

$$(T_p(x) \rightarrow x') \longrightarrow\!\!\!\rightarrow (x' \overset{Q}{=} x) \tag{4}$$

$$(T_+(x) \rightarrow x') \longrightarrow\!\!\!\rightarrow (x' \overset{Q}{>} x) \tag{5}$$

$$(T_-(x) \rightarrow x') \longrightarrow\!\!\!\rightarrow (x' \overset{Q}{<} x). \tag{6}$$

Our account of the development of IC rests heavily upon the assertion that, even for small collections of objects within the subitizing range, the system attempts to eliminate the final requantification by developing a reliable predictor of the outcome. Until recently (Mehler & Bever, 1967; Beilin, 1968; Gelman, 1972a) investigations of number conservation within the subitizing range have been eschewed, because they allegedly permit employment of a primitive perceptually based requantification of the transformed collection. Since this requantification obviates the need for conservation rules, collections of three or four elements cannot be used to study the development of conservation.

The validity of this line of argument can be questioned on two grounds. The account of Q_s presented earlier is inconsistent with the notion that small collections are quantified on a primitive, largely perceptual basis. (Such a notion is akin to the early "immediate apprehension" studies that lead to the discovery of subitizing.) Furthermore, the little empirical evidence available suggests the absence of any qualitative distinction in children's performance with collections above and below the subitizing range. Beilin (1968) presented a group of 3 to 5 yr-old children with an EC task employing two collections of four objects. The average score across all ages was 14% correct. A similar group of children were presented with a task involving only the terminal configurations in the original EC task. If children's responses were based upon direct requantification of the transformed collections, then no significant differences would be predicted between performance in the two groups. However, the second group, those that judged only final configurations, were correct on 37% of the trials. Thus, it appears that children in this age range are attempting to use rules that involve both initial conditions and transformations, although their rules are not yet correct. Additional support for this view, plus evidence that the rules are initially specific to the objects as well as the transformations is provided by Curcio, Robbins, and Ela (1971). They found that in a group of 167 preschoolers, 52 passed EC when their fingers were used as the items to be conserved, but only 13 passed when objects were used. Gelman (1972a) has convincingly demonstrated that young children possess number invariance rules, even though they cannot pass the classic conservation task. She further emphasizes the importance of decomposing the conservation task into distinct components, including both quantification operators and rules that utilize their results.

Our account of the discrimination of $T_{\pm}$ from T_p assigns a major role to the development and utilization of the subitizing quantification operator (Q_s). This seems plausible in the context of children's everyday experience. As Bunt (1951) has observed, in the course of play experiences and daily activities there is ample opportunity for children to employ Q_s to detect the consistent effects of addition, subtraction and perceptual transformations on small quantities of discrete items such as the other members of his family, toys, shoes, cutlery, and so on. With the development of Q_c and Q_e, this experience is gradually extended from small numbers of objects to larger and larger collections, progressing over time through the range usually studied in investigations labeled as conservation of discontinuous quantity, of number, and of continuous quantity.

In order to account for the development of conservation rules covering this extended range, we must add to the current capacity of our developing system. The additional capacity must include:

a. the ability to perform successfully on the equivalence conservation task.
b. the ability to conserve inequality as well as equality. Initial inequalities are of course impossible in the IC situation, but they are a regular feature of the child's experience in two-collection transformational situations.
c. the ability to conserve when comparisons are based upon the output of quantification operators other than Q_s.
d. the ability to coordinate pairs of quantitative symbols, as in conservation tasks involving the height and width of liquids in containers.

We will treat the development of each of these capacities in the order listed above, although we assume that they develop more or less concurrently once the components of identity conservation have developed. The only aspect of "stage" that we maintain is that the development of Q_s and the discrimination of $T_{\pm}$ from T_p *start* before the onset of any of the other capacities. This is analogous to what Flavell (1971) calls an extreme version of the gradual-development model.

> *It asserts that a stage-specific item achieves its final level of functional maturity only after and perhaps only well after the child has begun the development of the next stage's item . . . items from two or more stages can undergo developmental change concurrently* (*p. 427*).

Equivalence Conservation Based upon Subitizing

Consider the development of EC, assuming for the moment that Q_s is the only reliable indicator of quantity and that $T_\pm$ and T_p are differentiated. The general process of development is the same as that described in the development of IC. Systematic quantitative regularities among small collections of discrete objects are noticed and stored by the system. These particular aspects of environmental regularity are attended to by the system because they are among the few quantitative things that the system *can* notice at this stage of its development. Thus, for example, it notices that whenever two collections (within the Q_s range) are initially equal and T_p is performed on one of them, they remain equal. Similar regularities are generated by observation and storage of the effects of T_+ and T_- on initially equal collections. Through the process, described above, of storage, prediction and testing, and eventually prediction without testing, these three classes of regularities acquire the status of rules for EC:

$$(x \overset{Q}{=} y)(T_p(y) \to y') \longrightarrow\!\!\!\!\to (x \overset{Q}{=} y'), \tag{7}$$

$$(x \overset{Q}{=} y)(\mathrm{T}_+(y) \to y') \longrightarrow\!\!\!\!\to (x \overset{Q}{<} y'), \tag{8}$$

$$(x \overset{Q}{=} y)(\mathrm{T}_-(y) \to y') \longrightarrow\!\!\!\!\to (x \overset{Q}{>} y'). \tag{9}$$

EC (Rules 7–9) is clearly more difficult than IC (Rules 4–6) in that it requires an additional element in the condition and must distinguish three quantities (x, y, y') rather than two (x, x'). Furthermore, the extension of EC rules to *in*equivalence rules (see below) has no corresponding situation in IC, so that the two *systems* of rules are also of unequal difficulty. However, we do not assume a developmental sequence for EC and IC. Both develop concurrently once $T_\pm$ and T_p have been differentiated, and this differentiation is based upon regularities that occur in both paradigms. Evidence for the cooccurrence of IC and EC over a variety of materials and quantification operators has been reported by Moynahan and Glick (1972) and Northman and Gruen (1970).

We do not assume that EC operates by first applying a corresponding IC rule and then utilizing a transitive inference of the form:

$$(x \overset{Q}{=} y)(y \overset{Q}{=} z) \longrightarrow\!\!\!\!\!\rightarrow (x \overset{Q}{=} z).$$

There is no necessity for the prior existence of transitivity rules, since equivalence conservation is based upon empirical regularities.

In addition to the three regularities underlying equivalence conservation, the system also encounters the additional six forms of empirical results generated by each of the three transformations acting upon the two kinds of initial inequalities. Two of these six do not produce any regularities. When T_+ is applied to the lesser of two collections, or T_- to the greater, the result can be any of the three relations. The ambiguity arises from the lack of a metric upon T_+ and T_-, so that the *amount* of the transformation is not utilized in encoding the environmental regularities.

Since these two situations do not provide any consistent sequences, they do not become rules, because during the prediction-verification stage, the prediction will often be disconfirmed. Thus, only four additional rules develop from the empirical regularities involving initial inequalities:

$$(x \overset{Q}{>} y)(T_p(y) \rightarrow y') \longrightarrow\!\!\!\!\!\rightarrow (x \overset{Q}{>} y') \tag{10}$$

$$(x \overset{Q}{>} y)(T_-(y) \rightarrow y') \longrightarrow\!\!\!\!\!\rightarrow (x \overset{Q}{>} y') \tag{11}$$

$$(x \overset{Q}{<} y)(T_p(y) \rightarrow y') \longrightarrow\!\!\!\!\!\rightarrow (x \overset{Q}{<} y') \tag{12}$$

$$(x \overset{Q}{<} y)(T_+(y) \rightarrow y') \longrightarrow\!\!\!\!\!\rightarrow (x \overset{Q}{<} y'). \tag{13}$$

We have described the development of two sets of rules, both of which utilize knowledge elements that represent quantity. We have argued that the first operator to provide a basis for reliable quantitative comparisons (albeit at low numerical levels) is Q_s and that its early stabilization facilitates the discrimination of T_p, T_+, and T_-. These in turn facilitate the observation of the empirical regularities that underly the formation of the rules for EC and IC. In the course of this development of rules that utilize knowledge about quantity, the other two quantification operators are also developing, and it is to that development and its relation to IC and EC rules that we turn next.

Generalization of Conservation Rules to Q_c

The following discussion is confined to IC but it applies *mutatis mutandis* to EC. Once the empirical regularities based upon Q_s generate Rules 4–6, the system can attempt to apply these rules to those situations which are simultaneously within the range of both Q_s and Q_c. As Q_c becomes a reliable quantifier the rules are found to fit in the sense that the classes of transformations produce results *corresponding* to those in Rules 4–6. Initially the results are not *identical* to the earlier kinds of sameness and differentness, since in the case of Q_s they are derived from comparisons of lists of tolerance space symbols while in that of Q_c they are derived from comparisons based on the number series. Later, they probably are identical since, once number names are attached to the subitizing sublists, comparisons between quantitative values derived via Q_s can be carried out by means of the number series used by Q_c. Generalization of the rules to the operations of Q_c beyond the range of Q_s is based on a similarity of both the initial conditions—collections of discrete elements—and the transformations.

Three general points should be added here: (a) When Q_s and Q_e are considered, the fit stops at correspondence since Q_e results involve the comparison of size analog symbols. (b) Correspondences and/or identities between the outcomes of the quantification of discrete collections and of quantitative comparisons between collections obtained by using Q_s, Q_c, and Q_e may well be detected by the system in the course of applying any two operators to the same situation before, and quite independently of, any concern with the effects of transformations. Such preestablished correspondences would facilitate generalization of the rules. (c) The process of establishing correspondences between the outcomes of quantitative comparisons carried out via the three operators can be regarded as widening the semantic basis of verbal terms such as "more," "less," and "equal."

Generalization of Conservation Rules to Q_e

Most of the judgmental situations ultimately encountered by the child deal with materials that are either beyond the range of Q_c or Q_s or are continuous. In such cases only Q_e is appropriate. However, Q_e's widened scope has associated with it an increase in the complexity of the rules that utilize it.

The overall course of the generalization of conservation rules is the same when the quantitative symbols are generated by Q_e as by Q_c. First the three forms of transformations must be discriminated and associated with some regular outcomes. Then the classes of transformations are found to produce results corresponding to the conservation Rules 4–6. The use of Q_e introduces two difficulties. One is that the diversity of phenomena that must be represented as transformations is increased compared to the transformations upon small collections of discontinuous quantity. The other is that for the first time the system must quantify two dimensions of each quantity under consideration and the results of these double quantifications must be properly coordinated before any regularities can appear.

Inadequacy of unidimensional quantification. Initial attempts to quantify via Q_e lead to inconsistent results, because as Halford (1970) has indicated, the quantitative symbols generated for comparison are based upon only one dimension of the situation. In neither IC nor EC are there the requisite regularities among transformations and terminal judgment comparisons based upon unidimensional quantification. Any transformation can result in all three relations on a single dimension depending upon the change in the unattended dimension.

Consider the classic nonconservation responses obtained with the Piagetian EC task using discontinuous materials. Some of the children carry out a terminal comparison of quantitative symbols derived from the lengths of the two collections being compared. Others compare symbols representing the density or distance between the elements of the collections. The early appearance of such unidimensional quantification by Q_e is attested to by Descoeudres (1921). In a task requiring the construction of a row of objects equal in quantity to a row already constructed by the experimenter, the younger children pay less attention to the number to be reproduced than to the space to be occupied by the row of elements.

Two dimensional estimation. In its search for empirical regularities, the system begins to attend to both dimensions, and consistent relationships between transformations and terminal judgments begin to emerge. Table 1 represents the initial form of these regularities. The table shows the nine possible relational outcomes resulting from the transformation

TABLE 1

Initial Regularities for Identity and Equivalence Conservation Based upon Two-Dimensional Quantification Via Q_e

		Relation between x_e and x'_e or x_e and y'_e	
Line	Transformation	on Dimension 1	on Dimension 2
1	T_p	=	=
2	T_-	=	>
3	T_+	=	<
4	T_-	>	=
5	T_+, T_-, T_p	>	<
6	T_-	>	>
7	T_+	<	<
8	T_+	<	=
9	T_+, T_-, T_p	<	>

of initially equal collections. For example, the first line indicates that final equality on both dimensions is produced only by T_p. The second line indicates that equality on one dimension and a relatively smaller amount of the transformed collection on the other is produced only by T_-, etc. The table represents both EC (initially $x = y$ on both dimensions and the final test is on x and y') and IC (final test is on x' and x).

Two types of outcomes are equivocal with respect to the class of transformation that could produce them. Both of the outcomes that have simultaneous and opposite changes in two dimensions can be produced by T_p, T_+ or T_- (see lines 5 and 9 in Table 1). The other outcomes are uniquely associated with one of the three classes of transformations. Thus, the emerging system has two classification tasks facing it. First it must classify the types of transformation with respect to their effect upon quantity, and then it must resolve the ambiguity of the transformations that produced compensatory changes in two dimensions. Figure 2 illustrates the intersection of all possible pair-wise relationships for each transformation.

The solution to the first problem comes about through the association of some of the transformations in each class with corresponding transformations utilized in developing conservation rules based upon Q_s and Q_c. When utilizing two-dimensional quantification the system recognizes that the transformations associated with each of the seven unique outcomes in Table 1 are in many cases identical to the already developed transformational classifications. Although some of the transformations upon con-

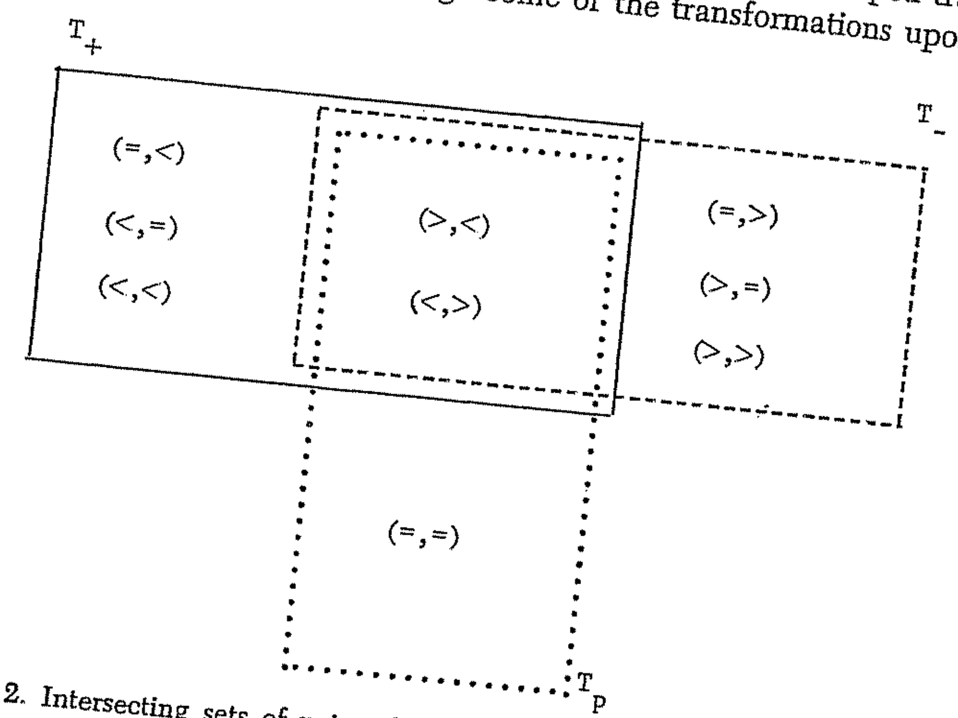

FIG. 2. Intersecting sets of pairs of relational outcomes for three classes of transformation.

tinuous quantity have no equivalent for discontinuous quantity, others are phenomenally similar enough for the association to be made. Furthermore, in those situations within the subitizing range, both Q_s and Q_e can be in use simultaneously. Thus, the same transformation will have an effect upon both the quantitative judgment based upon Q_s and the pair of relations (e.g., length, density) based upon Q_e. Having identified a specific transformation as one whose effect upon quantity is known via earlier rules, the system associates that quantitative outcome with the pair of quantitative outcomes based upon estimation.

The second problem involves the equivocal sequences in Table 1. Ultimately they are separated into regular sequences through the following process. Once the transformations yielding the seven unique outcomes have been discriminated into three categories, the appropriate relational attribute symbol (e.g., $x \overset{Q}{=} y$) is associated with the outcome of each transformation. The equivocal sequences then cease to be equivocal since they each comprise an initial situation and a transformation which has now been associated with a relational attribute symbol via one of the three categories. The appropriate relational attribute symbol is thus associated with each of these sequences.

The final form of the associations between pairs of quantitative comparison outcomes and relational attribute symbols is indicated in Table 2. All of them must be established for the complete generalization of the

TABLE 2
Final Regularities for Equivalence Conservation Based upon Two-Dimensional Estimation

Line	Transformation	Relation between x_e and y'_e on Dimension 1	Relation between x_e and y'_e on Dimension 2	Quantitative relation between x and y'
1	T_p	=	=	=
2	T_p	>	<	=
3	T_p	<	>	=
4	T_+	<	<	<
5	T_+	<	=	<
6	T_+	=	<	<
7	T_+	<	>	<
8	T_+	>	<	<
9	T_-	>	>	>
10	T_-	>	=	>
11	T_-	=	>	>
12	T_-	>	<	>
13	T_-	<	>	>

conservation rules to Q_e. Once this has taken place the appropriate relational attribute symbols can be predicted directly on the basis of the conservation rules, although children sometimes refer to the pairs of quantitative comparison outcomes in justifying their predictions. "They are still the same because this one is longer but that one is thicker."

An appreciation of the association between the transformations discriminated into three categories and the pairs of quantitative outcomes presumably underlies the capability to make the type of observation described by Halford (1970).

(a) *Whenever material is poured from A to B and nothing is added or subtracted, if there is any change in the material then there will be another change as well.*

(b) . . . *where a change occurs in one dimension, but not in any other, something will be added or subtracted.* (*p. 305.*)

Observation (a) is represented by lines 2 and 3 of Table 2 and observation (b) by lines 5, 6, 10, 11.

Once the process adumbrated has taken place in "common" situations, and the applicability of expanded conservation rules to the operations of Q_e has been established, the rules are applied to other situations involving continuous quantity. Such extensions follow the order of the phenomenal similarity of the transformations involved to those in the common situations. Thus, situations involving judgments and transformations on discontinuous quantity precede application to continuous quantity, (e.g., Smedslund, 1964) while within continuous quantity plasticene may be dealt with successfully prior to water (e.g., Uzgiris, 1964).

Why shift from one dimension to two? Piaget's (1957) illustration of equilibration provides an account of the transition to two-dimensional quantification in the context of a task requiring equivalence conservation of continuous quantity. He attributes the shift to three factors which produce an oscillation of attention between the dimensions of height and width and ultimately lead to the inclusion of both. First of all, attention tends to be directed to the dimension on which there is the greatest difference between the quantities of liquid. Since this dimension varies from trial to trial switching of attention between the dimensions is encouraged. Second, the same tendency is encouraged by the widely reported empirical finding that in a two alternative situation repeated responding to one of the alternatives increases the probability of an eventual response to the other. A final factor involved in the switch to a two dimensional basis is the child's dissatisfaction with the inconsistency of the outcomes of his unidimensional judgments.

Our explanation of the shift of Q_e to a two-dimensional mode lacks the probabilistic arguments addressed by Piaget, but it has a certain affinity with his emphasis on the child's dislike of inconsistency. In accordance with the redundancy elimination principle the system investigates the applicability to Q_e of the consistent sequences discovered in using Q_s. Independent of the redundancy elimination principle there is striking empirical evidence that the preservation of consistency and agreement between the judgmental outcomes of the Qs operates as an important goal. Intriguing examples of the expedients to which children will resort to obtain consistency between the outcomes of Q_e and Q_c have been provided by Inhelder and Sinclair (1969).

Attempts to apply the consistent sequences to Q_e and to preserve agreement between its outcomes and those of the other Qs are doomed to failure due to its undimensional basis. It is at this juncture in development that the second systemic principle becomes relevant. Confronted with the failed goal of detecting the conservation rules the system seeks to remove the difficulty by widening the range of the variables in the context being considered. This tactic gives rise to the quantification of the second dimension and to the resulting coordination process underlying the comparative judgments.

When the applicability to Q_c and Q_e of the conservation rules detected in the functioning of the Qs has been established the child has attained the criterion for "operational" conservation. The consistent sequences will be employed in making terminal judgments when any of the three operators are being employed. The adoption of this criterion has important implications. It leaves much more room for individual variations in the course of the development of conservation than, for example, the Genevan approach which tends to the view that the pace of development may vary but the sequence is the same for all. This emphasis on individual variation is consistent with a number of empirical findings. Greco (1962), for example, reports that some of his subjects when undergoing EC trials with two rows of discontinuous elements continue to give classic nonconservation responses on the terminal quantitative judgments but when questioned on the numerosity of the terminal collections are able to respond correctly that "You have six and I have six" or "There were five before." This distinction between *quotité* (number name) and *quantité* (numerical quantity) can be interpreted in terms of our criterion as stemming from subjects who are employing the consistent sequences in making terminal judgments when Q_c is in operation but whose Q_e is still functioning on a unidimensional and, thus, inconsistent basis.

Further evidence of the Q_c-Q_e sequence of development and of cases in which the reverse sequence appears to apply is provided by Churchill (1958). In an acceleration study she found that some of her subjects based their correct responses on tests of equivalence conservation of discontinuous quantity entirely on "numerical" features of the situation (Q_c) while others relied purely on "perceptual" features (Q_e). In a study aimed at tracing the course of individual development of conservation of discontinuous quantity, Wallace (1972) has also detected a number of threads of development which can be identified as Q_e-led or Q_c-led developmental sequences.

The model of "operational" conservation outlined is consistent with other aspects of the experimental evidence. As Beilin (1970) has pointed out in reviewing the many training studies stimulated by Piaget's work,

> "*What emerges from the data is the striking fact that a wide variety of methods, in fact, practically every type of experimental method leads to successful improvement in performance, even if not in every experiment.*" (*p. 45*)

This is precisely the type of outcome from acceleration studies which the model would predict. The developmental progression of the detection of the conservation rules by Q_s and the subsequent concurrent attempts to establish their relevance to Q_c and Q_e presents a broad front to training. Almost any of the acceleration treatments which have been employed to date could be shown to be likely to facilitate the development of children at particular points in the protracted and complex process of attaining "operational" conservation.

Production or prediction? The conservation rules described thus far all have the general form "if ⟨initial quantitative relation⟩ and ⟨transformation⟩ then ⟨final quantitative relation⟩." That is, they are predictors of the outcome of transformations. In the final form of conservation of continuous quantity, these predictions encompass two-dimensional compensatory relations. However, we have not described the development of an associated form of conservation "understanding": the production of appropriate transformations, given an existing relation and a desired one. For a system to be able to produce equivalence, it must have rules of the form "if ⟨current relation⟩ and ⟨desired relation⟩ then ⟨apply transformation *x*⟩." For every *predictive* rule of the form used in this paper there is a rule of the *productive* form that selects a transformation to achieve a desired goal. Existence of these pairs of rules in LTM is a functional requirement for the Genevan concept of reversibility.

Halford's (1970) description of specific "observations" has linked rules of the productive variety with some predictive rules. For example Halford says:

> *S can also learn that where nothing has been added or subtracted, the material can be returned to the original container where it will resume its original dimensions* (*p. 305*).

The representation of such knowledge would take the form of relatively powerful rules about the existence of classes of transformations with certain properties. For the example above the following type of rule would be required:

$$\text{if } T_p, \text{ then } \exists\, T_{p'}(y') \rightarrow y'' \text{ such that } \forall_i (y_i'' = y_i).$$

In words: "if a perceptual transformation T_p has occurred on collection y producing y', then there exists another transformation $T_{p'}$, that will produce an arrangement y'' such that the final dimensions of y'' will be equal to the original dimensions of y. This type of rule is an abstraction of the productive rules just described, and we would question its existence, in an explicit form, at the stage of concrete operations. However, the abstract rule is implicit in a system that contains a set of productive rules.

CONCLUSION

In this paper we have attempted to outline the development of some of the processes that underlie performance on conservation tasks. The model that emerges here consists of a precise description of pieces of a production system for various stages of development, and a less precise verbal description of some transition mechanisms that enable the system to move from stage to stage. In many areas the model is highly speculative because there is little empirical evidence to build upon; in other areas it is consistent with available evidence and it suggests some questions that can be resolved through further experimentation.

In concluding his own thorough analysis of the acquisition of conservation, Halford (1970) states

> *Such a [spontaneously functioning, self-regulating] system may be difficult to influence by experimental manipulation, because it is in its nature to compensate for any condition which is imposed. This may explain the curiously elusive quality of the underlying features of conservation performance, a feature which many investigators admit has puzzled them. There is probably no direct answer to the problem except that reasonably elaborate models should be continuously developed and refined to indicate the nature of the system . . . (p. 316).*

We concur with Halford, but caution that the aim of such model building efforts be not elaboration *per se*, but rather precision, with elaboration only as dictated by the demands of such precision. This has been the goal of our efforts. We believe that the limitations of the current statement can be eliminated only through more precise statements of both the stage and the transition mechanisms, and hence through even further elaboration.

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